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Waging Simple Wars

A Complete Characterization of Two Battlefield Blotto Equilibria

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Abstract We analyze the strategic allocation of resources across two contests as in the canonical Colonel Blotto game. In the games we study, two players simultaneously allocate their forces across two fields of battle. The larger force on each battlefield wins that battle, and the payoff to a player is the sum of the values of battlefields won. We completely characterize the set of Nash equilibria of all two battlefield Blotto games and provide the unique equilibrium payoffs. We also show how to extend our characterization to cover previously unstudied games with nonlinear resource constraints.

JEL codes: C72, H56, D7

Keywords Colonel Blotto game · Zero-sum game · Warfare · All-pay auction · Multi-unit auction

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1 Introduction

Since Borel (1921), Colonel Blotto games have been used to study a wide range of topics within systems defense,[1] conflicts and terrorism,[2] politics,[3] and auctions.[4] In Gross and Wagner's (1950) canonical paper, two players, Colonel Blotto and Enemy, are endowed with a quantity of B and E soldiers, respectively. They compete on m battlefields by simultaneously deciding how to allocate their soldiers across each. The player with more resources on a particular battlefield wins that battle, and a player's payoff is the sum of the values of the battlefields won.

Gross and Wagner (1950) found the unique equilibrium payoffs and some Nash equilibria of many Blotto games. Since then, much research has explored the implications of modifying their model.[5] Some study games where the payoffs change continuously, instead of discretely as one player outspends the other,[6] and others examine games that relax the constant sum assumption of the classic game.[7] Another line of research examines discrete Blotto games that require resource allocations to be integer valued.[8]

Roberson (2006) studied Blotto games with asymmetric resources ($B \neq E$) on three or more *equally valued* battlefields. For all but the most asymmetric of resource endowments, he characterized the univariate marginal distributions of players' mixed strategies in any Nash equilibrium and showed that they must be uniform. He also provided an equilibrium of each of the remaining "very asymmetric" Blotto games. Given these two results, he provided the unique equilibrium payoffs of any Blotto game on three or more equally valued battlefields. Gross (1950), Laslier (2002), and Thomas (2012) were able to provide equilibria and payoffs of canonical Blotto games with many battlefields ($m \geq 3$) of differing values, but symmetric resources ($B = E$). In this paper, we provide the first complete characterization of the set of Nash equilibria of all Blotto games with asymmetric resources *and* two battlefields of unequal value. Additionally, we extend our characterization to a class of previously unstudied Blotto games that allow for non-linear resource constraints. Characterizing the complete set of Nash equilibria determines all types of behavior we can (and cannot) expect in situations resembling this classic game.

We prove the completeness of our characterization, construct a continuous set of example equilibria, and provide a simple graphical algorithm for con-

[1] See Coughlin (1992), and Wu et al (2009)

[2] See Powell (2009), Powers and Shen (2009) and Hortala-Vallve and Llorente-Saguer (2010)

[3] See Colantoni et al (1975), Young (1978), Merolla et al (2005), Laslier and Picard (2002), Sahuguet and Persico (2006), and Le Breton and Zaporozhets (2010).

[4] Szentes and Rosenthal (2003) discuss this relationship. In general, the classic Blotto game is equivalent to a multi-unit, budget constrained, all-pay auction with no payoff for leftover resources.

[5] e.g. Kovenock and Roberson (2012a), and Adamo and Matros (2009).

[6] e.g. Blackett (1954), Blackett (1958), and Golman and Page (2009)

[7] e.g. Powell (2009), Kovenock and Roberson (2012b), and Roberson and Kvasov (2012).

[8] e.g.Hart (2008) and Dziubiński (2012)

structing equilibrium which conveys important intuition. Using our graphical algorithm, our results are easily pictured and interpreted. We place all two battlefield Blotto games in a "partition" $n \in \{1, 2, 3...\}$ based on the relative strength of players' endowments. As n increases, resource endowments become more equitable. In a Nash equilibrium to a game in partition n, each player plays a mixed strategy that randomizes over allocations from one of n disjoint sets of allocations. We determine the probability they play in each set, as well as provide the additional constraints on their strategies.

The organization for the remainder of this paper is as follows: Section 2 formally models the game. Section 3 characterizes the complete set of Nash equilibria, and section 4 provides a graphical algorithm used to construct the complete set of Nash equilibria. Section 5 concludes.

2 General Model

We consider the following generalization of Gross and Wagner's (1950) Blotto game on two battlefields. The two players simultaneously take action by setting $\{b_i\}_{i=1}^2$ and $\{e_i\}_{i=1}^2$, denoting Blotto and Enemy's respective strength of force on Battlefield i. If Blotto has a weakly stronger force on Battlefield i he wins that battle and receives a payoff on that battlefield of $w_i > 0$. Otherwise, Enemy wins that battle and receives w_i. The losing player receives a payoff of 0 from that battlefield. Players attempt to maximize the expected sum of their payoffs across the battlefields. A mixed strategy for Blotto or Enemy is a randomization over choices of $\{b_i\}_{i=1}^2$ or $\{e_i\}_{i=1}^2$, respectively. Therefore, we denote Blotto or Enemy's strategy as a probability measure μ_B or μ_E, respectively. Blotto's allocation is constrained by $b_1 \geq 0$, $b_2 \geq 0$ and a general resource constraint, $b_2 \leq f(b_1)$. So, his optimization problem is:

$$\max_{b_1,b_2} \left(\sum_{i=1}^{2} (Prob(b_i \geq e_i | \mu_E) \cdot w_i) \right) \quad s.t. \tag{1}$$

$$b_1 \geq 0,\ b_2 \geq 0,\ b_2 \leq f(b_1) \tag{2}$$

Similarly, Enemy is constrained by $e_1 \geq 0$, $e_2 \geq 0$, and $e_2 \leq g(e_1)$ and his optimization problem is:

$$\max_{e_1,e_2} \left(\sum_{i=1}^{2} (Prob(e_i > b_i | \mu_B) \cdot w_i) \right) \quad s.t. \tag{3}$$

$$e_1 \geq 0,\ e_2 \geq 0,\ e_2 \leq g(e_1). \tag{4}$$

The two battlefield games originally studied by Gross and Wagner (1950) are equivalent to the above with two exceptions.[9] First, they restrict the budget constraints to be linear by assuming $f(b_1) = B - b_1$ and $g(e_1) = E - e_1$ for

[9] They also normalize the payoffs differently to make the game zero-sum. We find this constant sum version more intuitive.

some constants B and E such that $B \geq E$. Second, they make a different assumption about payoffs when players allocate the same strength of force to a battlefield. We award ties to Blotto, the resource advantaged player, for consistency with the recent literature (Roberson, 2006). We define Blotto as resource advantaged by assuming that there is always some minimum distance between Blotto and Enemy's resource constraints or

$$\exists \zeta > 0 \; s.t. \; (\forall x_1, \; f(x_1) \geq g(x_1) + \zeta) \; and \; \left(\forall x_2, \; f^{-1}(x_2) \geq g^{-1}(x_2) + \zeta\right) \tag{5}$$

In Gross and Wagner's (1950) original linear game, this assumption implies that $B > E$. In non-linear games, Equation 5 rules out the possibility that the budget constraints touch at a point or cross.[10] In the special case of $B = E$ equilibria are easy to describe.[11]

We make some normalizations and assumptions to increase the clarity of the model. Without loss of generality, scale battlefield values to normalize $w_1 \equiv 1$. To ease notation, define $w \equiv \frac{w_2}{w_1}$. Also, assume the existence of finite intercepts of the resource constraints. Formally:[12]

$$\exists \; B_1, B_2 > 0 \; s.t. \; f(B_1) = 0, \; f(0) = B_2. \tag{6}$$

$$\exists \; E_1, E_2 > 0 \; s.t. \; g(E_1) = 0, \; g(0) = E_2. \tag{7}$$

Assume the resource constraints f and g are continuous, strictly decreasing functions. Also, we will assume that the resource constraints continue with slope -1 outside of quadrant I. As allocations on either battlefield may never be negative, this assumption does not actually affect the game, but eases the algebraic characterization.[13] Formally, we assume equations 8-11 :

$$f(x_1) = B_2 - x_1 \; \forall x_1 < 0 \tag{8}$$

$$f(x_1) = B_1 - x_1 \; \forall x_1 > B_1 \tag{9}$$

$$g(x_1) = E_2 - x_1 \; \forall x_1 < 0 \tag{10}$$

$$g(x_1) = E_1 - x_1 \; \forall x_1 > E_1 \tag{11}$$

[10] Often such games have simple Nash equilibria. For instance, if the budget constraints cross exactly once (or any odd number of times), each player can guarantee victory on exactly one battlefield. A simple Nash equilibrium is for each player to "hunker-down" and always send an unbeatable force to the battlefield on which they can guarantee victory.

[11] See Gross and Wagner (1950) for further details.

[12] We believe we could relax this assumption somewhat for Blotto. However, it seems realistic and makes the proof cleaner.

[13] We need not make such restrictive assumptions on the form of f and g outside of quadrant I. However, we chose this simple functional form since the assumption only affects the constraints where they cannot possibly bind.

3 Results

In this section we provide the complete characterization of the set of Nash equilibria of all two battlefield Blotto games. We make use of the following two composite functions of $f(\cdot)$ and $g(\cdot)$ (the resource constraints for Blotto and Enemy, respectively):

$$h(x) \equiv g^{-1}(f(x)) \tag{12}$$

$$p(y) \equiv g(f^{-1}(y)). \tag{13}$$

Note that we follow the standard convention where $h^0(x) \equiv x$ (and similarly for other functions). If necessary, $h^{-1}(x) \equiv f^{-1}(g(x))$ and $p^{-1}(y) \equiv f\left(g^{-1}(y)\right)$ as one would expect. Then, the ith iterates of $h(x)$ and $p(y)$ can be defined as follows: $h^{i+1}(x) \equiv h(h^i(x))$ and $p^{i+1}(x) \equiv p(p^i(x))$. Similarly, the $-i$th iterates of $h(x)$ and $p(y)$ can be defined as follows: $h^{-i-1}(x) \equiv h^{-1}(h^{-i}(x))$ and $p^{-i-1}(x) \equiv p^{-1}(p^{-i}(x))$. Also, note that functions 12 and 13 are well-defined for all x and y given equations 8-11 and the fact that $f(\cdot)$ and $g(\cdot)$ are both strictly decreasing functions. Thus, $p^i(\cdot)$, $p^{-i}(\cdot)$, $h^{-i}(\cdot)$ and $h^i(\cdot)$ are all well-defined strictly increasing functions $\forall i = 0, 1, 2, ...$

The intuition behind h and p is as follows. $h(\cdot)$ takes a Battlefield 1 allocation x, and returns an Enemy Battlefield 1 allocation $h(x)$ such that the following is true: If Blotto and Enemy allocate x and $h(x)$ to Battlefield 1, respectively, and choose a resource constraint binding Battlefield 2 allocation ($f(x)$ and $g(h())$, then they will have the same Battlefield 2 allocation. $p(\cdot)$ takes a Battlefield 2 allocation y and returns an Enemy Battlefield 2 allocation $p(y)$ such that the following is true: If Blotto and Enemy allocate y and $p(y)$ to Battlefield 2, respectively, and choose a resource constraint binding Battlefield 1 allocation, then they will have the same Battlefield 1 allocation.

We now describe a partitioning of the set of possible two battlefield Blotto games (not player action spaces) based upon the players' exogenous resource constraint functions. Figure 1 shows examples of Blotto games in partitions 1, 2, and 3 respectively. Intuitively, we number the partitions based on the number of times it is possible to reflect one of the end points of Enemy's resource constraint, E_1 or E_2, off of Blotto's resource constraint and his own in quadrant I, as shown in Figure 1.

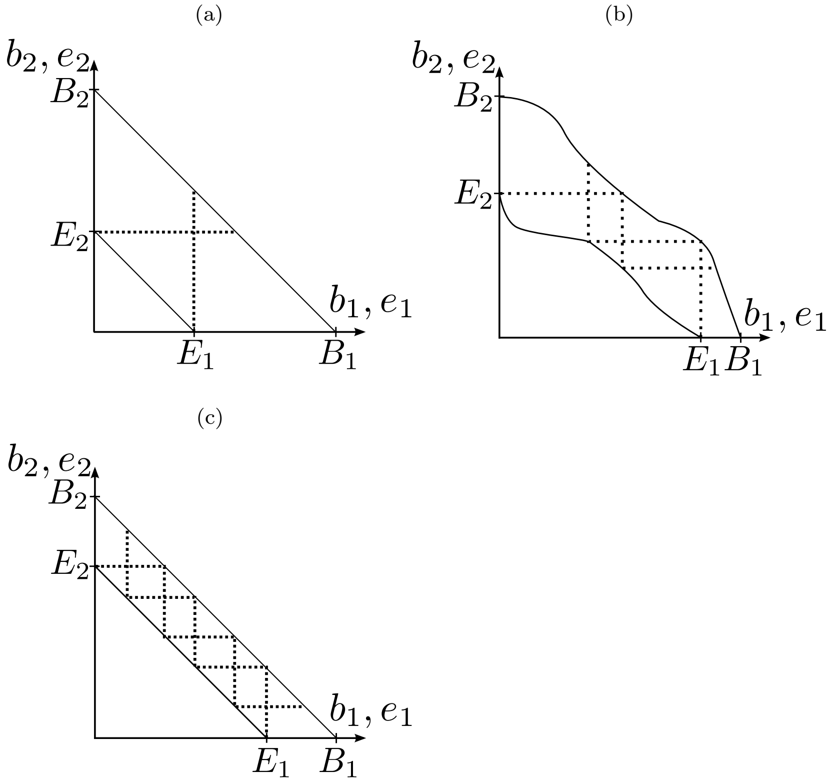
Remark 1 Every Blotto game satisfying the conditions in Section 2 falls into one partition n, where the strictly positive integer n satisfies the following condition:

$$E_2 \in (f\left(h^{n-2}(E_1)\right), f\left(h^{n-1}(E_1)\right)]. \tag{14}$$

Proof For any such game, Equation 14 must be true for exactly one strictly positive n. Consider an integer $i \geq 0$. By Equation 5, $f\left(g^{-1}\left(f\left(h^{i-1}(E_1)\right)\right)\right) \geq g\left(g^{-1}\left(f\left(h^{i-1}(E_1)\right)\right)\right) + \zeta$ for some $\zeta > 0$. Simplifying we have

$$f\left(h^i(E_1)\right) \geq f\left(h^{i-1}(E_1)\right) + \zeta. \tag{15}$$

Fig. 1: Blotto and Enemy Resource Constraints for Blotto Games in Partitions 1, 2, and 3.



As $f\left(h^{-1}\left(E_1\right)\right) = 0$ by definition, iterating on equation 15 implies that $f\left(h^i\left(E_1\right)\right) \geq \zeta i$. So, for $i \geq \frac{E_2}{\zeta}$ we have $f\left(h^i\left(E_1\right)\right) \geq E_2$. Given that $f\left(h^i\left(E_1\right)\right)$ is strictly increasing in i (by Equation 15) and that there is some i for which $f\left(h^i\left(E_1\right)\right) \geq E_2$, Equation 14 must be true for exactly one integer n.[14]

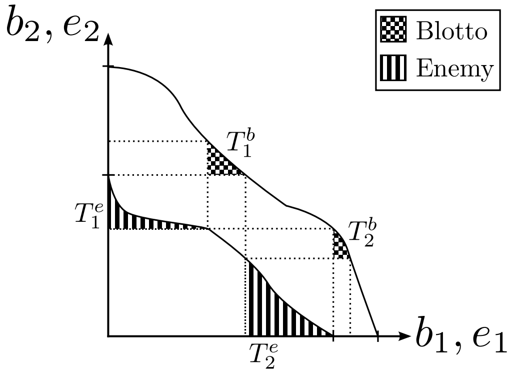
Consider a two battlefield Blotto game in partition $n \geq 2$. Define the following sets of allocations:

$$\forall i = 1, ..., n \quad T_i^b \equiv \{(b_1, b_2):\ (b_1 \geq h^{n-i}(E_1),\\ b_2 \geq p^{i-1}(E_2),\ b_2 \leq f(b_1)\} \tag{16}$$

$$\forall i = 2, 3, ..., n-1 \quad T_i^e \equiv \{(e_1, e_2):\ (e_1 > f^{-1}\left(p^{i-2}\left(E_2\right)\right),\\ e_2 > f\left(h^{n-i-1}\left(E_1\right)\right),\ e_2 \leq g(e_1))\} \tag{17}$$

[14] Note that one could provide a similar definition and explanation in terms of $f^{-1}\left(p^i\left(E_2\right)\right)$.

Fig. 2: The T_i's of a Game in Partition 2



$$T_1^e \equiv \{(e_1, e_2) : \ (e_1 \geq 0, \ e_2 > f\left(h^{n-2}\left(E_1\right)\right), \ e_2 \leq g(e_1))\} \tag{18}$$

$$T_n^e \equiv \{(e_1, e_2) : \ (e_1 > f^{-1}\left(p^{n-2}\left(E_2\right)\right), \ e_2 \geq 0, \ e_2 \leq g\left(e_1\right))\} \tag{19}$$

Equations 18 and 19 are implied by 17 with the exception of weak inequalities at the zero bounds. Note that if $E_2 = f\left(h^{n-1}\left(E_1\right)\right)$, which occurs at a partition boundary, then all the T_i^b's will contain only one point. For example, in T_1^b we would have one point, $(b_1, b_2) = \left(h^{n-1}\left(E_1\right), f\left(h^{n-1}\left(E_1\right)\right)\right) = \left(f^{-1}\left(E_2\right), E_2\right)$. Figure 2 graphs the T_i's for a game in partition 2.

We now define sets that will be useful in specifying the restrictions on how players can randomize over their T_i's. $\forall i < 1, 2, ..., n-1, \forall x \in \mathbb{R}$:

$$j_b^{x,i} \equiv \{(b_1, b_2) : \ ((b_1, b_2) \in T_i^b, \ b_1 < x)\} \tag{20}$$

$$k_b^{x,i} \equiv \{(b_1, b_2) : \ ((b_1, b_2) \in T_{i+1}^b, \ b_2 \geq g\left(x\right))\} \tag{21}$$

$$j_e^{x,i} \equiv \{(e_1, e_2) : \ ((e_1, e_2) \in T_{i+1}^e, \ e_1 \leq x)\} \tag{22}$$

$$k_e^{x,i} \equiv \{(e_1, e_2) : \ ((e_1, e_2) \in T_i^e, \ e_2 > f\left(x\right))\} \tag{23}$$

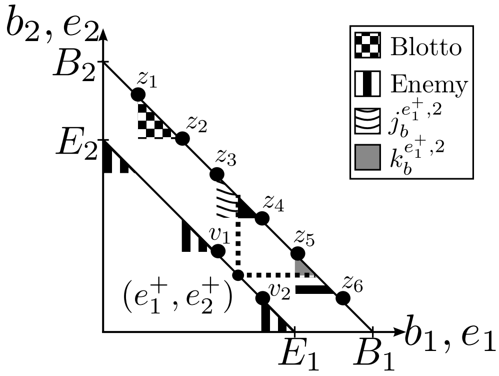
If Enemy were to play $(x, g(x))$, $j_b^{x,i}$ is the set of allocations in T_i^b where Blotto would lose on Battlefield 1, and $k_b^{x,i}$ is the set of allocations in T_{i+1}^b where Blotto would win on Battlefield 2. Conversely, if Blotto were to play $(x, f\left(x\right))$, $j_e^{x,i}$ is the set of allocations in T_{i+1}^e where Enemy would lose on Battlefield 1, and $k_e^{x,i}$ is the set of allocations in T_i^e where Enemy would win on Battlefield 2.

Now, define Ω^B as the set of probability measures, μ_B, which satisfy the following two properties:

Property 1b: $\forall i = 1, ..., n$

$$\mu_B(T_i^b) = \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$$

Fig. 3: Property 2b



Property 2b: $\forall i < 1, 2, ..., n-1, \quad \forall x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-1}\left(E_2\right)\right)]$:

$$\mu_B(j_b^{x,i}) \leq \mu_B(k_b^{x,i}) \cdot w$$

Property 1b specifies how Blotto must allocate his probability mass to each of the T_i^b's. For instance, in the equally weighted battlefield case, Blotto must play in each of the T_i^b's with probability equal $\frac{1}{n}$. Property 2b provides restrictions on how Blotto may allocate his probability mass within the T_i^b's. These restrictions ensure that any Enemy full expenditure deviation is not payoff improving. Figure 3 illustrates Property 2b for a Blotto game in Partition 3.

Now, define Ω^E as the set of probability measures, μ_E, with the following two properties:

Property 1e: $\forall i = 1, ..., n$

$$\mu_E(T_i^e) = \frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}$$

Property 2e: $\forall i = 1, 2, ..., n-1, \quad \forall x \in \left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), h^{n-i-1}(E_1)\right)$

$$\mu_E(j_e^{x,i}) \leq \mu_E(k_e^{x,i}) \cdot w$$

Property 1e specifies how Enemy must allocate his probability mass to each of the T_i^e's. For instance, in the equally weighted battlefield case, Enemy must play in each of the T_i^e's with probability equal $\frac{1}{n}$. Property 2e provides restrictions on how Enemy may allocate his probability mass within the T_i^e's.

Given these properties we provide three theorems which we prove in Appendix A:

Theorem 1 *When Blotto plays a strategy $\mu_B \in \Omega^B$ and Enemy plays a strategy $\mu_E \in \Omega^E$, Blotto's expected payoff is*

$$\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j} \tag{24}$$

and Enemy's expected payoff is

$$\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}. \tag{25}$$

Proof See Appendix A.1

Given that this game is constant sum, these payoffs will be the unique equilibrium payoffs (see Lemma 9 in Appendix A.3).

Theorem 2 *Any pair of strategies* $\{\mu_B, \mu_E\}$ *such that* $\mu_B \in \Omega^B$ *and* $\mu_E \in \Omega^E$ *constitute a Nash equilibrium.*

Proof See Appendix A.2.

In other words, pairing a Blotto strategy from Ω^B, with an Enemy strategy from Ω^E forms a Nash equilibrium. Note that in general there will be a continuum of Blotto (Enemy) strategies, μ_B (μ_E), which form an equilibrium with any corresponding opponent strategy.[15]

Theorem 3 *The complete set of Nash Equilibria of any two battlefield Colonel Blotto game is the set of pairs* $\{\mu_B, \mu_E\}$ *such that* $\mu_B \in \Omega^B$ *and* $\mu_E \in \Omega^E$.

Proof See Appendix A.3.

Property 1b(e) requires that Blotto (Enemy) randomizes over each of his T_i^b's (T_i^e's) with probability $\frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j}$, $\left(\frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}\right)$. Property 2b(e) simply requires that Blotto (Enemy) distributes his mass over the areas from Property 1b(e) in such a way that he does not provide his opponent with any profitable deviations.

Theorem 4 *There are an uncountably infinite number of equilibria of any two battlefield Blotto game.*

Appendix B proves Theorem 4. In the proof we show how to take any continuous univariate cumulative distribution function (CDF), with range $[0,1]$ over the domain $\left[h^{n-1}(E_1), f^{-1}(E_2)\right]$ $\left(\left[0, h^{n-1}(E_1)\right)\right)$ as given and then construct a unique bivariate Blotto (Enemy) equilibrium strategy distribution from that CDF. So long as $\left[h^{n-1}(E_1), f^{-1}(E_2)\right]$ $\left(\left[0, h^{n-1}(E_1)\right)\right)$ is not a scalar, there are an uncountably infinite number of such CDFs.[16] Since it is impossible for

[15] See Appendix B for a method to construct such a continuum of equilibria.

[16] For example, consider the set of univariate CDFs of the form:

$$\left(\frac{x - h^{n-1}(E_1)}{f^{-1}(E_2) - h^{n-1}(E_1)}\right)^a$$

$\forall a \in [1,2]$. Any CDF in this set has range $[0,1]$ over the domain $\left[h^{n-1}(E_1), f^{-1}(E_2)\right]$. There are an uncountably infinite number of such CDFs. So, using the method described in Appendix B, we could generate an uncountably infinite number of equilibrium bivariate Blotto strategies from this set of univariate CDFs.

$\left[0, h^{n-1}(E_1)\right)$ to be a scalar, at least Enemy has an uncountably infinite number of equilibrium strategies. There are then an uncountably infinite number of equilibria of any Blotto game.

Note that there are additional equilibrium strategies beyond those that can be constructed as discussed in Appendix B. Since the processes used in Appendix B assume the CDFs of battlefield 1 allocations in $\left[h^{n-1}(E_1), f^{-1}(E_2)\right]$ and $\left[0, h^{n-1}(E_1)\right)$ are continuous, they don't produce equilibrium strategies with mass points. In Appendix A.3 we make use of equilibrium Blotto and Enemy strategies with mass points in T_1^b and T_1^e, respectively.

4 Nash Equilibrium General Construction

We now provide an algorithm which completely characterizes the set of Nash equilibria graphically. Until Section 4.4 we assume Gross and Wagner's (1950) original resource constraints, $f(b_1) = 1 - b_1$ and $g(e_1) = E - e_1$ for some $E < 1$,[17] and that the battlefields are equally valuable, $w \equiv 1$.

Section 4.1 graphically constructs the set of all Nash equilibria in the trivial case where $E \leq \frac{1}{2}$. Section 4.2 constructs the more complicated set of all Nash equilibria in the next partition of the game where $\frac{1}{2} < E \leq \frac{2}{3}$. Section 4.3 shows how the graphical algorithm extends to other games.

4.1 The Trivial Case: Partition 1, $E \leq \frac{1}{2}$

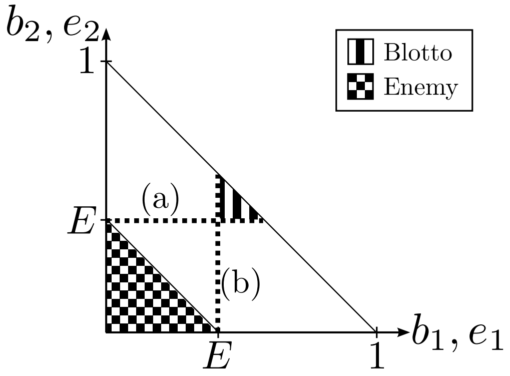
If Colonel Blotto has twice the forces of Enemy, he can guarantee himself victory on both battlefields by deploying at least E on both battlefields, as shown in Figure 4. Clearly, a Nash equilibrium is formed by any (feasible) Blotto strategy that always sends at least E to both battlefields paired with any (feasible) Enemy strategy. Against such Blotto strategies Enemy can expect a payoff no greater than 0, and no matter what, Blotto can expect a payoff no greater than 2. Note that when $E = \frac{1}{2}$ Blotto's T_1^b collapses to a point; he must always play (E, E) in equilibrium.

4.2 Partition 2, $\frac{1}{2} < E \leq \frac{2}{3}$

Unlike in Section 4.1, Blotto no longer has enough soldiers to guarantee victory on both battlefields. However, he has enough resources to do the following: choose a battlefield by the flip of a fair coin; send E soldiers to that battlefield and at least $\frac{E}{2}$ to the other battlefield. Against such a strategy, Enemy can only send more than half his forces (for example all his forces) to one battlefield. Enemy hopes to mismatch; he hopes to deploy the majority of his forces to the battlefield to which Blotto sent at least $\frac{E}{2}$ (in which case he will win one battle). Blotto hopes to match. If they both send a larger force to the same

[17] Technically, Gross and Wagner (1950) set $f(b_1) = B - b_1$. Here we normalize $B = 1$.

Fig. 4: Resource Constraints for a (Trivial) Blotto Game in Partition 1



battlefield, Blotto wins both. Intuitively, in any Nash equilibrium, each player heavily attacks one battlefield and sends a smaller force to the other battlefield.

Consider this game graphically in Figure 5. Figure 5a shows that Blotto has two separate areas where he may play while attacking one battlefield heavily. Line (a) shows that he must play more than E on Battlefield 2 if he is to ensure victory there. Line (b) shows a similar condition when he decides to attack Battlefield 1 heavily. If Blotto plays in each of these areas with probability $\frac{1}{2}$, Enemy should expect the same payoff from heavily attacking either battlefield.

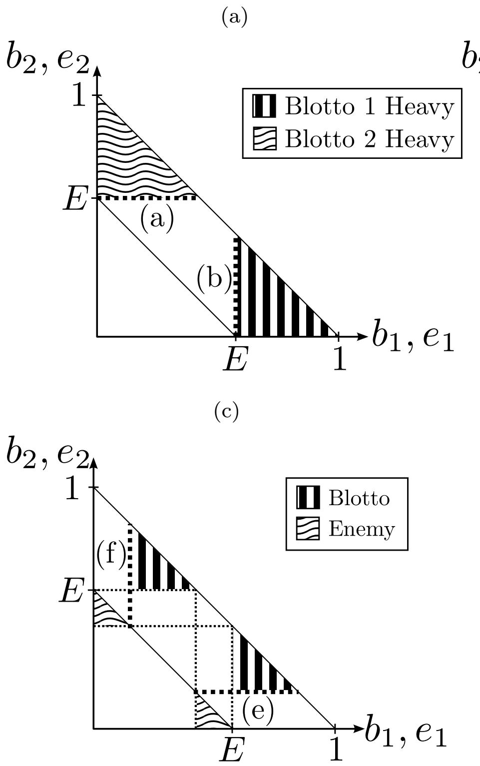
Observe Figure 5b. The horizontal line (c) shows that when Enemy is attacking Battlefield 2 heavily, he needs to be sure his force is large enough to beat any residual force Blotto might send to Battlefield 2 when Blotto is attacking Battlefield 1 heavily.[18] The vertical line (d) shows a similar condition when Enemy attacks Battlefield 1 heavily. If Enemy plays in each of the two wavy areas with equal probability, Blotto does not know which battlefield to attack heavily, nor which only needs a smaller force.

Consider Figure 5c. The horizontal line (e) shows that when Blotto attacks Battlefield 1 heavily, he needs to send enough forces to Battlefield 2 to ensure victory there if Enemy attacks Battlefield 1 heavily, but still sends a small force to Battlefield 2. The vertical line (f) demonstrates a similar condition when Blotto attacks Battlefield 2 heavily. We now have two areas for both Blotto and Enemy, and we know they must play in both areas with equal probability. Note that when $E = \frac{2}{3}$ Blotto's two areas will be points.

Properties 2b and 2e from Section 3 dictate how players should distribute mass within those two areas. In any Nash equilibrium Blotto expects a payoff of $1\frac{1}{2}$ and Enemy expects $\frac{1}{2}$. The distribution of mass within the two areas cannot provide the other player with a higher expected payoff if they deviate. We demonstrate these conditions in Figure 6 in terms of the j's and k's defined in equations 20-23. Figure 6a demonstrates how certain Blotto distributions

[18] Recall that Enemy can never win on Battlefield 2 when Blotto is also attacking Battlefield 2 heavily.

Fig. 5: $\frac{1}{2} < E \le \frac{2}{3}$ Construction

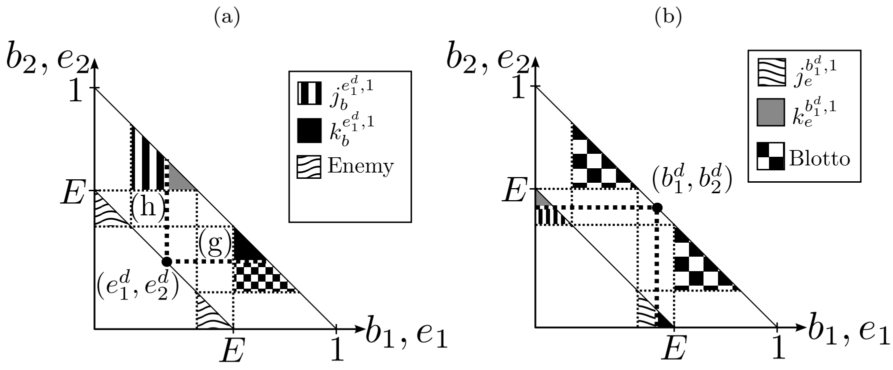


could provide Enemy with profitable deviations from his prescribed strategy. Observe the potential deviation (e_1^d, e_2^d). Line (h) divides the area where Blotto is attacking Battlefield 2 heavily into two parts; in $j_b^{e_1^d,1}$, $b_1 < e_1^d$. Line (g) similarly divides the area where Blotto is attacking Battlefield 1 heavily; in $k_b^{e_1^d,1}$, $b_2 \ge e_2^d$. We know that Enemy can only expect a payoff of at most $\frac{1}{2}$ if he were to deviate here.

Relative to playing in T_1^e, playing (e_1^d, e_2^d) increases Enemy's chances of winning on Battlefield 1 by Blotto's frequency of play in area $j_b^{e_1^d,1}$, but decreases Enemy's chances of winning on Battlefield 2 by Blotto's frequency of play in area $k_b^{e_1^d,1}$. Therefore, the total mass Blotto plays over area $j_b^{e_1^d,1}$ can be no more than the mass over area $k_b^{e_1^d,1}$, as stated in Property 2e. Figure 6b demonstrates how similar restrictions affect Enemy's potential distributions.

Notice that we chose (e_1^d, e_2^d) (and (b_1^d, b_2^d)) arbitrarily. Any full expenditure deviation is possible. Thus there is a continuum of such restrictions (one for each full expenditure deviation) on how Blotto and Enemy can randomize over the two areas. If there are no full expenditure payoff improving deviations, then

Fig. 6: $\frac{1}{2} < E \leq \frac{2}{3}$ Mass Restrictions



there are no payoff improving deviations as both players' payoffs are weakly increasing in allocations to either battlefield.

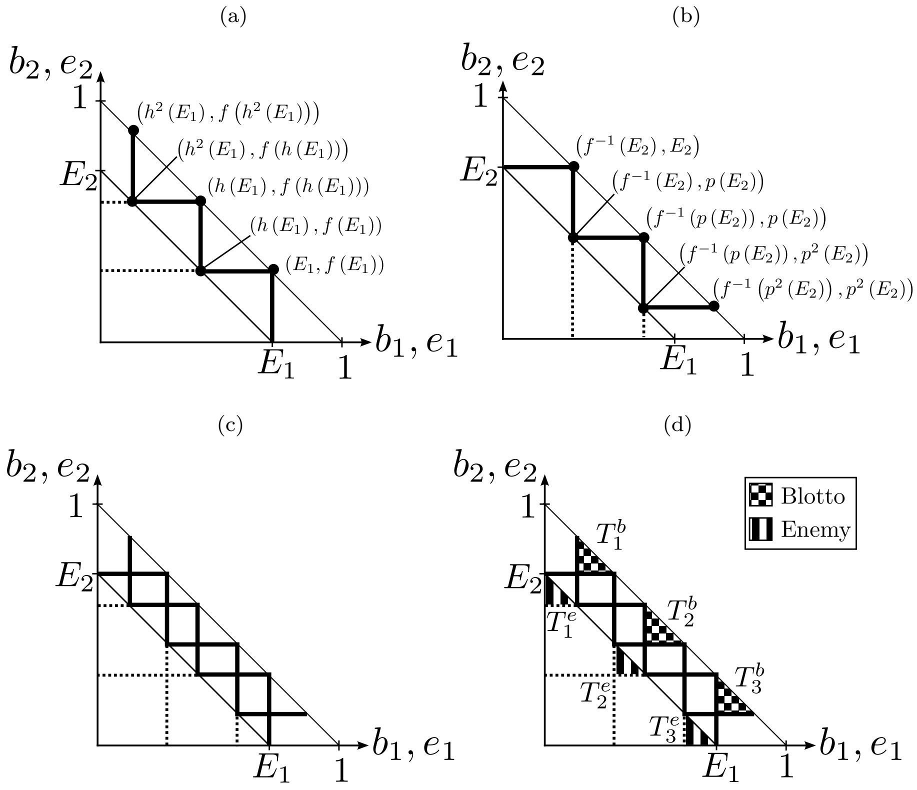
We can describe the complete set of Nash equilibrium to Blotto games in partition 2 graphically as follows: the set of Blotto and Enemy randomizations that place $\frac{1}{2}$ of each players' mass on the respective player's two areas shown in Figure 5c, and distribute the mass within those two areas in such a way that their opponent has no profitable deviations (as shown in Figure 6).

4.3 The General Approach

In the previous Section, Blotto could randomize between $(E, \frac{E}{2})$ and $(\frac{E}{2}, E)$; each battlefield was always allocated at least $\frac{E}{2}$. When $E > \frac{2}{3}$ Blotto no longer has a sufficient resource advantage to support those allocations. Thus, he now must be concerned that Enemy might split his forces across both battlefields (e.g. playing $\left(\frac{E}{2}, \frac{E}{2}\right)$). We will now provide our graphical algorithm to determine the set of Nash equilibrium in such cases. We provide an illustration of our method for Partition 3 where $E \in (\frac{2}{3}, \frac{3}{4}]$ in Figure 7. For the rest of this section, we will use the notation E_1 and E_2 to ease later comparisons to the game with general resource constraints.

The graphical algorithm is a three step process. In step one, as shown in Figure 7a, we draw a (solid) vertical line coming out of the point $(E_1, 0)$. Every time this line intersects a resource constraint we rotate it 90 degrees (alternating counterclockwise, clockwise, counterclockwise...). Observe in Figure 7a how implementing two of these 90 degree rotations corresponds to subjecting an allocation to the h function defined in equation 12. We stop the rotations once we reach a point (b_1, b_2) on Blotto's resource constraint where $b_2 \geq E_2$. Every time the solid line intersects Enemy's resource constraint we also draw a dotted line showing how the solid line would continue if we did not rotate it.

Fig. 7: The General Graphical Method



A game's partition as defined in Section 3 is equivalent to the number of times step one intersects Blotto's resource constraint. For example, the game depicted in Figure 7a is in partition $n = 3$ as we intersected Blotto's resource constraint three times.

Step two is similar to step one. As shown in Figure 7b, we draw a (solid) horizontal line coming out of the point $(0, E_2)$. Again, when the line intersects a resource constraint we rotate it 90 degrees (alternating clockwise, counter-clockwise, clockwise...). Observe in Figure 7b how implementing two of these 90 degree rotations corresponds to subjecting an allocation to the p function defined in equation 13. We stop the rotations once we reach a point (b_1, b_2) on Blotto's resource constraint where $b_1 \geq E_1$. We again draw a dotted line showing how the solid line would continue every time it intersects Enemy's resource constraint. After doing both steps one and two we will have a graph like Figure 7c.

In step three we use our graph to find the regions over which each player will randomize. After completing steps one and two we have n triangles directly below Enemy's resource constraint and $2n-1$ triangles directly below Blotto's resource constraint as shown in Figure 7c.[19] We label the triangles directly below Enemy's resource constraint $T_1^e, ..., T_n^e$, from top left to bottom right as in Figure 7d. These will be the areas over which Enemy can randomize in equilibrium. For the triangles directly below Blotto's resource constraint we label the top left T_1^b. As we move down and right along the resource constraint we skip the next triangle and then label T_2^b, skip another, label T_3^b and so on until we reach T_n^b. We can see this in Figure 7d.

We stop steps one and two when we reach points on Blotto's resource constraint where $x_2 \geq E_2$ and $x_1 \geq E_1$, respectively. If we stop step one because $x_2 = E_2$ then we will necessarily stop step two because $x_1 = E_1$ and the lines drawn by these steps will overlap exactly and each T_i^b will consist of a single point.

In equilibrium, each player will play in each of their T_i's with probability $\frac{1}{n}$. Therefore, with probability $\frac{1}{n}$ they will "match," or, for some j, Blotto and Enemy will play allocations in T_j^b and T_j^e, respectively. In this case, Blotto wins both battlefields. Otherwise (with probability $\frac{n-1}{n}$) they will "mismatch," and Blotto and Enemy will play allocations in T_i^b and T_j^e, respectively, such that $i \neq j$. In this case each player wins one battlefield. Thus, for any allocation in one of their T_i's, Blotto expects a payoff of $\frac{n+1}{n}$ while Enemy expects $\frac{n-1}{n}$.

In order to completely characterize the set of equilibrium strategies, we need to discuss additional restrictions on how players can randomize within their T_i's. These restrictions are very similar to the restrictions shown earlier in Figure 6; they prevent profitable deviations by the opponent. Since either player's expected payoff is weakly increasing in allocations on either battlefield, ensuring that there are no full expenditure, expected payoff increasing deviations is sufficient.

Properties 2b and 2e preventing profitable deviations apply to partition n in the same manner as they did in partition 2. This is illustrated in Figure 8.

In Appendices B and A.3 we provide formal examples of Blotto and Enemy strategies that would satisfy our characterization. However, for now the reader may wish to consider the following Nash equilibrium in terms of the game presented in Figure 8: Each player plays in each of their T_i's with probability $\frac{1}{n}$ by playing that mass uniformly over their resource constraint within each T_i. Note that since the slope of both resource constraints is -1, the conditions preventing payoff improving deviations will always hold with equality in this equilibrium.

4.4 Generalizations of the Graphical Algorithm

In this section we demonstrate how to extend our graphical algorithm to apply to Colonel Blotto games with asymmetric battlefield weights and/or non-linear resource constraints.

[19] We are not discussing any of the larger triangles in the graph which contain smaller shapes (e.g. the axes and the resource constraints form triangles, but these are not what we are interested in). The triangles we are discussing are empty in Figure 7c

Fig. 8: Region 3 Mass Restrictions

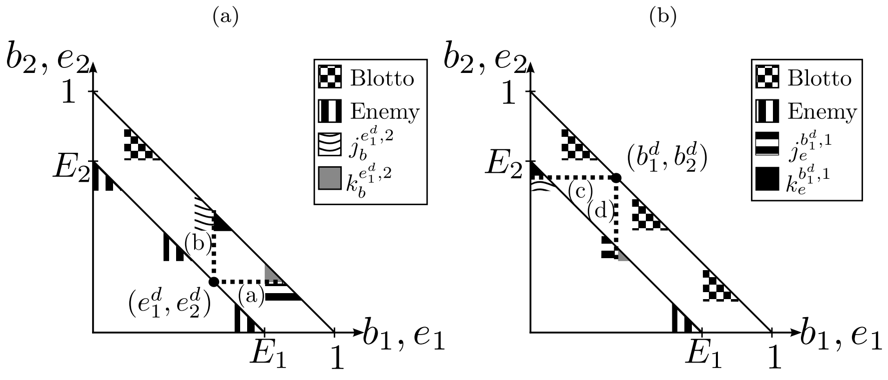
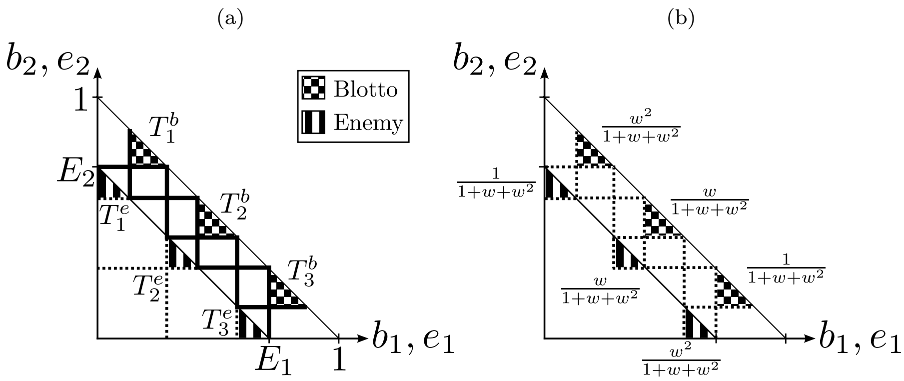


Fig. 9: Weights



4.4.1 Asymmetric Battlefields

Now we relax our earlier assumption that both players care about both battlefields equally; we no longer restrict $w_2 = w_1$. Previously, players would play in each T_i with probability $\frac{1}{n}$. Now that the players place different weights on the two battlefields, they need to play in each T_i with a different probability in order to make their opponent indifferent between his own T_i's. While we will explain how to graphically characterize the set of equilibrium strategies generally, Figure 9 shows how our process applies in partition 3. We follow the process in Section 4.3 to find the T_i's over which players may randomize (Figure 9a).

Consider the expected payoff to Enemy of playing in some T_i^e compared to playing in T_{i+1}^e (e.g. T_1^e v.s. T_2^e in Figure 9a). The realized payoff of playing in either will be the same unless Blotto plays in T_i^b or T_{i+1}^b. Moving from T_{i+1}^e to T_i^e allows Enemy to now win Battlefield 2 if Blotto plays in T_{i+1}^b but at the cost of now losing on Battlefield 1 when Blotto plays in T_i^b. Since Enemy values Battlefield 2 w times as much as Battlefield 1, his added chance of losing Battlefield 1 needs to be w times his added chance of winning Battlefield 2. This implies that $\mu_B(T_i^b) = w \cdot \mu_B(T_{i+1}^b)$. As this must hold for all i:

$$\mu_B(T_i^b) = \frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j},$$

These weights are shown in Figure 9b. Similar logic gives the probability Enemy will play in each T_i^e. The results are shown in Figure 9b.

We still need to place additional restrictions on how players can randomize within their T_i's. However, we can show the restrictions in the same manner as before in Figure 8. The only difference is that now we need to account for the different weights. This is easily reconciled. In terms of Figure 8a the restriction now becomes $\mu_B(j_b^{e_1^d,2}) \leq \mu_B(k_b^{e_1^d,2}) \cdot w$ and similarly for μ_E. The set of equilibria in these non-constant sum Blotto games is going to be the set of pairs of Blotto and Enemy strategies that satisfy the conditions we've just described.

4.4.2 Force Effectiveness and Non-Linear Resource Constraints

In many situations resource constraints may be non-linear. Perhaps each player faces different (dis)economies of scale for different levels of resources allocated to each battlefield. In order to allow for this generalization, we relax the assumptions made earlier on the resource constraints and allow for any strictly decreasing $g(\cdot)$ and $f(\cdot)$ which satisfy Equations 5-7. We believe there are no prior solutions to this generalization of the Blotto game.

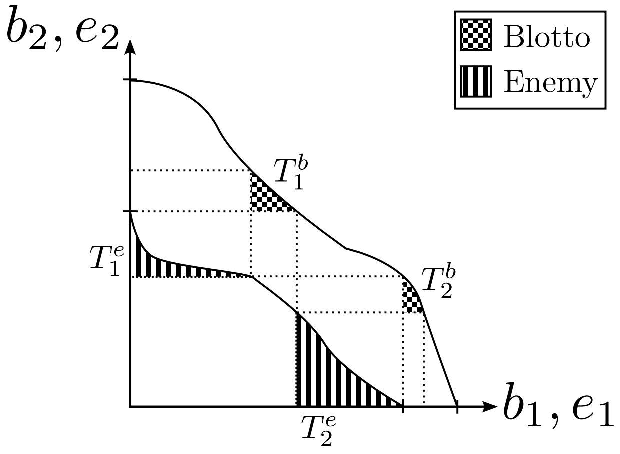
Our previous graphical algorithm described in Section 4.3 works here without modification. We only need to use non-linear resource constraints. Figure 10 demonstrates how our method works with non-linear constraints in the same way as Figure 7 did for the case of linear resource constraints of slope -1. Since we are allowing non-linear budget constraints the T_i's may no longer be triangles, but we retain the notation T. To allow for this modification and general battlefield values, the analysis from Section 4.4.1 carries over directly.

5 Conclusion

We provide a complete characterization of the set of Nash equilibria of any canonical two battlefield Blotto game and for Blotto games with non-linear resource constraints. Furthermore, we provide a method for constructing continuous sets of equilibria of any such game, and provide an algorithm for illustrating the equilibrium strategies graphically.

Experimentally, Chowdhury et al (2013) confirm most of Roberson's (2006) theoretical predictions, and Arad and Rubinstein (2012) use an experiment in

Fig. 10: General Budget Constraints



a Colonel Blotto environment to study alternative decision making processes. Prior to our work, testing two battlefield Blotto predictions (experimentally or empirically) would have been problematic. For the set of games we study, the previous literature has provided only some equilibria, but the set of Nash equilibrium strategies we find is infinite. Suppose we observed players in an environment like ours using strategies that differed from any of the previously known Nash equilibrium strategies. How would we know if they were failing to play Nash strategies in general, or simply playing an unknown Nash strategy? Knowing the set of equilibria solves this issue.

Gross and Wagner (1950) show that equilibrium payoffs for two battlefield Blotto games can change discontinuously with changes in players' initial linear resource constraints. Our work may allow for equilibrium payoffs to change similarly as the shape of players' non-linear resource constraints change. This may yield fruitful discussions regarding endogenous resource constraints, (either levels of resources or shapes of constraints) similar to the way that Kovenock et al (2010) analyze endogenous battlefield dimensionality. Just as the disadvantaged player may be willing to pay to increase the dimensionality of the battlefields in Kovenock et al (2010), players may be willing to pay to "reshape" their resource constraints.

As noted earlier, Roberson (2006) considers Blotto games on three or more equally valued battlefields ($m \geq 3$) with asymmetric resources ($B \neq E$). For most such games he characterized the unique univariate marginals of players' strategies, showing that they were uniform. However, for all other "very asymmetric" ($B \in [(m-1)E, mE]$) games he provided an equilibrium. These games are fundamentally different; the strategies he provides do not have uniform marginals. All non-trivial two battlefield Blotto games meet the "very asymmetric" criteria ($B \in [E, 2E]$).

In the strategies Roberson (2006) provides for very asymmetric games, each player randomly chooses two battlefields. Then, they each randomly choose one of k allocations to the two battlefields to play with probability $\frac{1}{k}$. These are reminiscent of the n T_i's played with probability $\frac{1}{n}$ that we characterize for two battlefield games with symmetric battlefield values. This suggests that our complete characterization may extend to cover the "very asymmetric" Blotto games on many battlefields to which Roberson (2006) provides an equilibrium. Our allowance for asymmetric battlefield values may carry over as well. Further extensions of our work could lead to the first characterization of the set of Nash equilibria of canonical Blotto games with asymmetric resources and more than two unequally valued battlefields.

Our work should prove useful for further empirical and theoretical analysis. While the set of Nash equilibria we find is infinite, our characterization provides some easily testable criteria: In equilibrium each player should only play allocations from one of n distinct areas. We define these areas and provide the probability with which they should play in each. We demonstrate the theoretical utility of our approach by characterizing the set of equilibria of a previously unsolved generalization of the Blotto game allowing for non-linear resource constraints. Our results provide fertile ground for further work, especially considering the relationship between our characterization and the very asymmetric games analyzed by Roberson (2006).

Characterizing the complete set of Nash equilibrium strategies of the canonical Blotto game exposes new equilibria and the full set of conditions constraining them. Interestingly, these conditions do not necessarily require either player to fully expend their resources, even though there is no payoff for unallocated resources. This suggests possible equilibrium refinement techniques or potential insight into the shadow value of the players' resources. Extending the logic of our Nash construction algorithm yields insights into more complicated variants of the game, which may be more representative of real military, political, or other environments.

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A Proofs of Theorems 1-3

First, we prove that pairs of strategies from Ω^B and Ω^E provide the expected payoffs from Theorem 1. Then we prove that all pairs of strategies from Ω^B and Ω^E constitute a Nash equilibrium. Finally, we show that no other strategies are a part of any Nash equilibrium.

First, define two projection operators. $\Psi_1(S)$ is the set of all scalars that are the first dimension of some two dimensional point in the set S, the set of Battlefield 1 allocations given a set S of two dimensional battlefield allocations.

$$\Psi_1(S) \equiv \{x_1 : (\exists x_2 \in \mathbb{R} \; s.t. \; (x_1, x_2) \in S)\}$$

$\Psi_2(S)$ is defined similarly:[20]

$$\Psi_2(S) \equiv \{x_2 : (\exists x_1 \in \mathbb{R} \; s.t. \; (x_1, x_2) \in S)\}$$

We also define the set of all points in some T_i^b (T_i^e):

$$T^b \equiv T_1^b \cup T_2^b \cup ... \cup T_n^b$$

$$T^e \equiv T_1^e \cup T_2^e \cup ... \cup T_n^e$$

Recall the definitions of h and p from Equations 12 and 13. Lemma 5 gives us the intervals of battlefield allocations within the various T_i's.

Lemma 5 $\forall i = 1, 2, ..., n :$

$$\Psi_1(T_i^b) = [h^{n-i}(E_1), f^{-1}\left(p^{i-1}\left(E_2\right)\right)], \tag{26}$$

$$\Psi_2(T_i^b) = [p^{i-1}(E_2), f\left(h^{n-i}\left(E_1\right)\right)]. \tag{27}$$

$\forall i = 2, 3, ..., n-1 :$

[20] For instance, if $S = \{(1, 3), (2, 5)\}$, then $\Psi_1(S) = \{1, 2\}$ and $\Psi_2(S) = \{3, 5\}$.

$$\Psi_1(T_i^e) = (f^{-1}\left(p^{i-2}\left(E_2\right)\right), h^{n-i}(E_1)), \tag{28}$$

$$\Psi_2(T_i^e) = (f\left(h^{n-i-1}\left(E_1\right)\right), p^{i-1}(E_2)), \tag{29}$$

For $i = 1$ we have the following:

$$\Psi_1(T_1^e) = [0, h^{n-1}(E_1)), \tag{30}$$

$$\Psi_2(T_1^e) = (f\left(h^{n-2}\left(E_1\right)\right), E_2], \tag{31}$$

and for $i = n$:

$$\Psi_1(T_n^e) = (f^{-1}\left(p^{n-2}\left(E_2\right)\right), E_1], \tag{32}$$

$$\Psi_2(T_n^e) = [0, p^{n-1}(E_2)), \tag{33}$$

The last four equations are implied by equations 28 and 29, but vary due to the closed bounds at the extremes. We will simultaneously prove Lemma 5 with our proof of Lemma 6.

First, informally we could rewrite the interval $\left[0, f^{-1}\left(p^{n-1}\left(E_2\right)\right)\right]$ as

$$[\Psi_1(T_1^e), \Psi_1(T_1^b), \Psi_1(T_2^e), \Psi_1(T_2^b), ..., \Psi_1(T_n^e), \Psi_1(T_n^b)]$$

or we could rewrite the interval $\left[0, f\left(h^{n-1}\left(E_1\right)\right)\right]$ as

$$[\Psi_2(T_n^e), \Psi_2(T_n^b), \Psi_2(T_{n-1}^e), \Psi_2(T_{n-1}^b), ..., \Psi_2(T_1^e), \Psi_2(T_1^b)].$$

More formally,

Lemma 6 $\Psi_1(T^e) \cup \Psi_1(T^b) = \left[0, f^{-1}\left(p^{n-1}\left(E_2\right)\right)\right]$ *and* $\Psi_2(T^e) \cup \Psi_2(T^b) = \left[0, f\left(h^{n-1}\left(E_1\right)\right)\right]$ *while* $\Psi_1(T^e) \cap \Psi_1(T^b) = \{E_1\}$ and $\Psi_2(T^e) \cap \Psi_2(T^b) = \{E_2\}$.

Proof Refer to equations 16-19. Consider the bounds for any $e_1 \in \Psi_1(T_i^e)$. Its open (closed when $i = 1$) infimum is $f^{-1}\left(p^{i-2}\left(E_2\right)\right)$. Changing the two other constraints on T_i^e to equalities and solving we find that for $e_1 \in \Psi_1(T_i^e)$ the open (closed when $i = n$) supremum is $h^{n-i}(E_1)$, which is the closed infimum of $b_1 \in \Psi_1(T_i^b)$. Similar algebra for the other relevant bounds in Lemma 5 and iteration from $i = 1$ confirms Lemmas 5 and 6.

Based on this we know that $\forall i \in \{1, ..., n\}$ any $e_1 \in \Psi_1(T_i^e)$ is strictly less than any $b_1 \in \Psi_1(T_i^b)$, which is strictly less than any $e_1 \in \Psi_1(T_{i+1}^e)$. The former inequality is weak when $i = n$. (Obviously, we ignore the latter when $i = n$). Also, $\forall i \in \{1, ..., n\}$ any $e_2 \in \Psi_2(T_{i-1}^e)$ is strictly greater than any $b_2 \in \Psi_2(T_i^b)$ which is strictly greater than any $e_2 \in \Psi_2(T_i^e)$. The later inequality is weak when $i = 1$. Obviously, we ignore the former when $i = 1$. More formally we have:

Lemma 7

$\forall i \in \{1, ..., n-1\}$,

$$\left((e_1^i \in \Psi_1(T_i^e),\ e_1^{i+1} \in \Psi_1(T_{i+1}^e),\ b_1^i \in \Psi_1(T_i^b)) \implies (e_1^i < b_1^i < e_1^{i+1})\right) \tag{34}$$

$$(e_1^n \in \Psi_1(T_n^e), b_1^n \in \Psi_1(T_n^b)) \implies e_1^n \le b_1^n \tag{35}$$

$\forall i \in \{2, 3, ..., n\}$,

$$\left((e_2^{i-1} \in \Psi_2(T_{i-1}^e),\ e_2^i \in \Psi_2(T_i^e),\ b_2^i \in \Psi_2(T_i^b) \implies (e_2^i < b_2^i < e_2^{i-1})\right) \tag{36}$$

$$(e_2^1 \in \Psi_2(T_1^e), b_2^1 \in \Psi_2(T_1^e)) \implies e_2^1 \le b_2^1 \tag{37}$$

Proof Lemma 7 follows directly by examining the bounds in Lemma 5.

A.1 Proof of Theorem 1

Recall **Theorem 1**: When Blotto plays a strategy $\mu_B \in \Omega^B$ and Enemy plays a strategy $\mu_E \in \Omega^E$, Blotto's expected payoff is $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$ and Enemy's expected payoff is $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$.

Proof Given Lemma 7, we know that against any Blotto strategy in Ω^B, when Enemy plays in T_i^e his probability of winning on Battlefield 1 is:

$$\mu_B(T_1^b \cup ... \cup T_{i-1}^b) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$$

His probability of winning Battlefield 2 is:

$$\mu_B(T_{i+1}^b \cup ... \cup T_n^b) = \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}.$$

The total expected payoff is then:

$$\frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j} + w \cdot \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j} = \frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j} \tag{38}$$

for any allocation in (any) T_i^e.

Similarly, against any Enemy strategy from above, when Blotto plays in T_i^b his probability of winning Battlefield 1 is:

$$\mu_E\,(T_1^e \cup ... \cup T_i^e) = \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$$

His probability of winning Battlefield 2 is:

$$\mu_E\,(T_i^e \cup ... \cup T_n^e) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$$

His total expected payoff is then

$$1 \cdot \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j} + w \cdot \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j} = \frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j} \tag{39}$$

for any allocation in (any) T_i^b.

A.2 Proof of Theorem 2

In this section we prove that any pair of strategies $\{\mu_B, \mu_E\}$, such that $\mu_B \in \Omega^B$ and $\mu_E \in \Omega^E$, in fact forms a Nash equilibrium. Before proceeding with the formal proof, we provide the intuition. Properties 1b and 1e specify that in any equilibrium Blotto and Enemy each randomize over n distinct areas ($T_1^b, ..., T_n^b$ and $T_1^e, ..., T_n^e$). Blotto and Enemy's potential equilibrium allocations on either battlefield only overlap at one point in the following sense:

$$\Psi_1(T^b) \cap \Psi_1(T^e) = \{E_1\},$$

$$\Psi_2(T^b) \cap \Psi_2(T^e) = \{E_2\}.$$

Though in the figures it may appear as though all boundaries should be included in these sets, careful inspection of the boundary conditions shows that the boundaries for Enemy are open, and closed for Blotto. Therefore, they should not in fact be included in these sets.

Given that ties always go to Blotto, we calculated players' expected payoffs in Appendix A.1. When they both play strategies satisfying Properties 1b and 1e, Blotto achieves an expected payoff of $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$ while Enemy earns $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$. We will show that given these payoffs, Property 2b(e) ensures that Enemy (Blotto) has no full expenditure allocation which provide a payoff strictly greater than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$ $(\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j})$. Since all allocations in the players' supports provide the same payoff, and there exist no allocations providing higher payoffs, pairs of strategies from these distributions constitute a Nash equilibrium. We now move to the formal proof.

Recall **Theorem 2:** *Any pair of strategies $\{\mu_B, \mu_E\}$ such that $\mu_B \in \Omega^B$ and $\mu_E \in \Omega^E$ constitute a Nash equilibrium.*

Proof We show that there are no allocations for Enemy or Blotto that provide a strictly higher expected payoff than their payoffs from Theorem 1. Note that if either player were to have an expected payoff improving deviation from the strategies we defined, they must have a full expenditure payoff improving deviation, as the expected payoffs must be weakly increasing in strength on either battlefield. Therefore, we only need to show that there are no payoff improving full expenditure deviations. So, we check full expenditure deviations outside of any T_i^e or T_i^b.

Consider a full expenditure Enemy deviation $e^d = (e_1^d, e_2^d),\ e^d \notin T^e$.[21] Given the bounds on the T_i^e's and Lemma 5, e^d must lie "between" some T_i^e and T_{i+1}^e in the following sense: $\forall e_1^i \in \Psi_1(T_i^e), e_1^{i+1} \in \Psi_1(T_{i+1}^e) \quad e_1^i < e_1^d < e_1^{i+1}$, and similarly for e_2^d. Let (e_1, e_2) be an allocation in T_i^e. Examine Property 2b with $x = e_1^d$. Given Lemma 7, the realized payoff to Enemy of playing e^d against any of our Blotto strategies will be the same as if he had played (e_1, e_2) unless Blotto plays in T_i^b or T_{i+1}^b. If Blotto plays in T_i^b, the deviant allocation *may* do better[22] on Battlefield 1 (without changing the outcome on Battlefield 2). The cost is that if Blotto plays in T_{i+1}^b the deviant strategy may do worse on Battlefield 2 (without changing the outcome on Battlefield 1). Using the notation of Property 2b, any b_1 in $j_b^{e_1^d,i}$ will lose to e_1^d (while it would have beat e_1) and any b_2 in $k_b^{e_1^d,i}$ will beat e_2^d (while it would have lost to e_2). Property 2b then says that by moving from any (e_1, e_2) in T_i^e to (e_1^d, e_2^d), the additional probability of winning on Battlefield 1 is weakly less than the additional probability of losing on Battlefield 2 times the weight placed on that battlefield. Therefore, no full expenditure deviation (e_1^d, e_2^d) is payoff improving, and therefore no deviation is payoff improving.

The same line of reasoning applies directly to Property 2e and full expenditure deviations by Blotto which lie "between" some T_i^b and T_{i+1}^b. Specifically, Property 2e ensures that a full expenditure deviating allocation by Blotto, (b_1^d, b_2^d), cannot be payoff improving. Simply set $b_1^d = x$ in the property and the same line of reasoning follows. Additionally, there are full expenditure deviations which do not lie "between" some T_i^b and some T_{i+1}^b (e.g. $(B, 0)$ and $(0, B)$). Specifically, there are two more deviating types of full expenditure allocations: a (b_1^d, b_2^d) where $\forall (b_1, b_2) \in T_1^b \quad b_1^d < b_1 \quad \textit{and} \quad f\left(b_1^d\right) = b_2^d > b_2$ or a $(b_1^{\#}, b_2^{\#})$ where $\forall (b_1, b_2) \in T_n^b \quad b_1^{\#} > b_1 \quad \textit{and} \quad f\left(b_1^{\#}\right) = b_2^{\#} < b_2$. In the former, Blotto increases allocations to Battlefield 2 at the expense of Battlefield 1, relative to T_1^b. However, in T_1^b, Blotto is guaranteeing victory on Battlefield 2, so this can not be payoff improving. Similar logic applies to the later type of allocations. Given Lemma 5, and Lemma 7, there are no other types of full expenditure deviations.

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[21] Clearly $e_2^d = g(e_1^d)$.

[22] By "do better" on Battlefield 1 we mean e_1^d would be strictly greater than Blotto's Battlefield 1 allocation, whereas e_1 would be weakly less.

Thus, if Blotto plays $\mu_B \in \Omega^B$ and Enemy plays $\mu_E \in \Omega^E$, they would both be playing best responses to the other's strategy. Therefore any such pair $\{\mu_B, \mu_E\}$ constitutes a Nash equilibrium.

A.3 Proof of Theorem 3

We now prove that there are no uncharacterized strategies which could be part of a Nash equilibrium. Before proceeding, we will need two lemmas that hold for any two player constant sum game.

Consider a two-player constant-sum game where Player 1 chooses a strategy $x \in X$ and Player 2 chooses a strategy $y \in Y$. Let $f^i(x, y)$ denote the expected payoffs to Player i when Player 1 plays x and Player 2 plays y. Also, let Ξ^i denote the set of player i strategies that are a part of some Nash equilibrium.

Definition 8 *A game is said to feature constant payoffs if, for each player, the expected payoff is the same in each Nash equilibrium.*

Lemma 9 *If a Nash equilibrium exists for a two-player constant sum game, that game features constant payoffs. In other words,* $\exists \{c_i\}_{i=1}^2$ *such that in every Nash equilibrium* $\{x_j, y_j\}$*,* $f^i(x_j, y_j) = c_i$ *for all* $i = 1, 2$.

For a formal proof of the preceeding and proceeding lemmas see Vorob'ev (1977, pp. 1-10).

Lemma 10 ***Equilibrium Interchangeability***

If a Nash equilibrium exists to a two-player constant sum game, then every strategy that a player uses in any Nash equilibrium forms a Nash equilibrium with any opponent strategy from any (other) Nash equilibrium. In other words: for all $x^* \in \Xi^1$ *and all* $y^* \in \Xi^2$*,* $\{x^*, y^*\}$ *constitutes a Nash equilibrium.*

Equilibrium Interchangeability and constant payoffs are useful when proving the completeness of our characterization.[23] Equilibrium Interchangeability allows us to consider equilibrium strategies for Blotto and Enemy separately. Unlike with most multiple equilibria games, there is no need to worry about pairing with a particular opponent equilibrium strategy. If we discover just one Nash equilibrium (pair of strategies), all the remaining equilibria are simply the cross of all the Blotto strategies that form an equilibrium with the one known Enemy strategy, and all the Enemy strategies that form an equilibrium with the one known Blotto strategy.

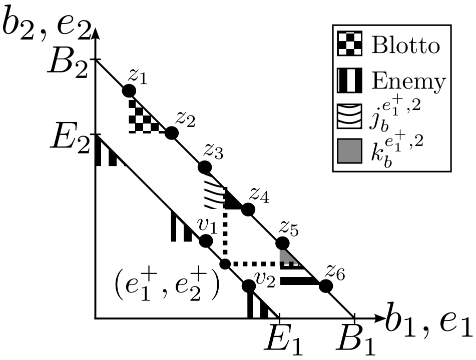
Our proof of the completeness of our characterization proceeds as follows. We first prove that all Enemy strategies that are a part of some Nash equilibrium are in Ω^E. Then we prove that all Blotto strategies that are a part of some equilibrium are in Ω^B. Therefore, the set of all Nash equilibria is the set of pairs of strategies from Ω^E and Ω^B by Equilibrium Interchangeability.

In the proof, we make use of the strategy μ_B^* where in each T_i^b Blotto plays the allocations $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$ and $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$ with probability $\frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j}$ each. These points correspond to $z_1 - z_6$ in Figure 11 for a game in region 3. Blotto never plays any other allocations. Recall that if $E_2 = f\left(h^{n-i}(E_1)\right)$ each of the T_i^b's contain only one allocation, $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right) = \left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$. In this case this single allocation is played with probability $\frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j}$.

In each T_i^b, $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$ and $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$ are the intersections of Blotto's resource constraint with the two other bounds on T_i^b. So, Property

[23] We are not the first to use these two results when analyzing Blotto Games (e.g. see Roberson (2006)).

Fig. 11: An Example



1b holds as $2 \cdot \frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j} = \frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j} = \mu_B^*\left(T_i^b\right)$. Now consider Property 2b. As Blotto is playing $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$ in T_i^b with probability $\frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j}$, $\mu_B^*(j_b^{x,i}) \leq \frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j} - \frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j} = \frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j}$, $\forall i < 1, 2, ..., n-1, \quad \forall x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-1}\left(E_2\right)\right)]$. As g is strictly decreasing, the minimum of $\mu_B^*(k_b^{x,i})$ for $x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-1}\left(E_2\right)\right)]$ occurs when $x = h^{n-i}(E_1)$. Then $\forall x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-1}\left(E_2\right)\right)]$, $\mu_B^*(k_b^{x,i}) \geq \frac{w^{n-(i+1)}}{2\cdot\sum_{j=0}^{n-1} w^j}$ as $g\left(h^{n-i}(E_1)\right) = f\left(h^{n-i-1}(E_1)\right)$ and $\left(h^{n-(i+1)}(E_1), f\left(h^{n-(i+1)}(E_1)\right)\right)$ is played with probability $\frac{w^{n-(i+1)}}{2\cdot\sum_{j=0}^{n-1} w^j}$ in T_{i+1}^b. Therefore $\mu_B^*(k_b^{x,i}) \cdot w \geq \frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j} \geq \mu_B^*(j_b^{x,i})$ and Property 2b holds, $\mu_B^* \in \Omega^B$.

Lemma 11 *Any Enemy strategy which is a part of some Nash equilibrium is in Ω^E.*

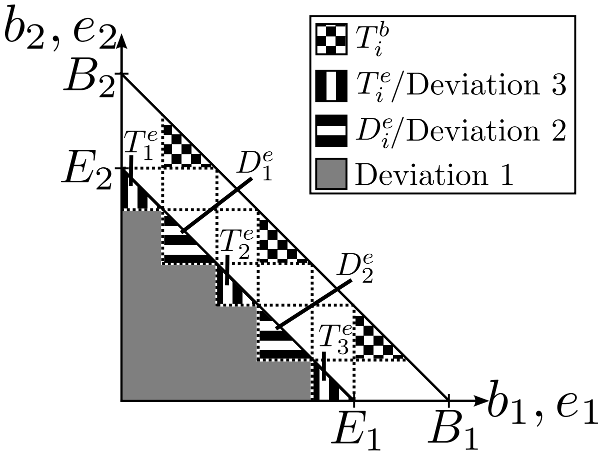
Proof We prove Lemma 11 by contradiction. Suppose there exists a Nash equilibrium Enemy strategy that is not in Ω^E. Such a strategy must then either violate Property 1e or satisfy Property 1e and violate Property 2e. In proving that all our strategies were indeed part of a Nash equilibrium, we have already shown how a violation of Property 2e alone would provide Blotto with a payoff improving deviation, so we rule out that possibility. The only other way Lemma 11 could be false is if there were a Nash equilibrium Enemy strategy which violated property 1e. We divide deviations from property 1e into three possible cases. Figure 12 provides a graphical reference (in Region 3) to aid the reader.

Intuitively, **Deviation 1** represents Enemy placing some mass on allocations which, relative to some T_i^e, always send less to one battlefield, without increasing the allocation to the other. **Deviation 2** represents Enemy placing mass on allocations which, relative to any T_i^e always send less to one battlefield, but increase the allocation to the other. **Deviation 3** has him playing an "incorrect" mass on some T_i^e (i.e. $\mu_E(T_i^e) \neq \frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}$).

Deviation 1: Enemy could play over an area that sends less to both battlefields than some (e_1, e_2) in some T_i^e. Formally, this would have Enemy play a $\mu_E^{d_1}$ such that the following three statements hold for some feasible set of Enemy allocations S:[24]

[24] Feasibility implies: $\forall (e_1, e_2) \in S \; e_1 \geq 0, \; e_2 \geq 0, \; e_2 \leq g\left(e_1\right)$.

Fig. 12: Enemy Deviations in Region 3



$$S \cap T^e = \emptyset \tag{40}$$

$$\mu_E^{d_1}(S) > 0 \tag{41}$$

$$\forall (e_1^d, e_2^d) \in S, \quad \exists (e_1, e_2) \in T^e \quad s.t. \quad e_1^d \leq e_1 \quad and \quad e_2^d \leq e_2 \tag{42}$$

Condition 40 implies that at least one of the two inequalities in 42 always holds strictly.

Suppose **Deviation 1** holds for some S and consider an allocation $(e_1^d, e_2^d) \in S$. Without loss of generality, suppose $\exists\,(e_1, e_2) \in T_i^e$ such that $e_1^d \leq e_1$ and $e_2^d < e_2$. Either $e_2^d < e_2^\#$ for all $e_2^\# \in \Psi_2\left(T_i^e\right)$ or not. If so, (e_1^d, e_2^d) cannot be a best response to μ_B^* which has Blotto playing Battlefield 2 allocations equal to the open lower bound of $e_2^\# \in \Psi_2\left(T_i^e\right)$ with positive probability. Enemy could increase his expected payoff by playing (e_1, e_2). Alternatively, there could exist an $e_2^\# \in \Psi_2\left(T_i^e\right)$ for which $e_2^d \geq e_2^\#$. Since we know $e_2^d < e_2 \in \Psi_2\left(T_i^e\right)$ we must have that $e_2^d \in \Psi_2\left(T_i^e\right)$. Given the bounds of T_i^e and equation 40, it must be the case that $e_1^d \leq f^{-1}\left(p^{i-2}\left(E_2\right)\right)$ the open lower bound of $\Psi_1\left(T_i^e\right)$, a Battlefield 1 allocation Blotto plays with positive probability in μ_B^*. Therefore (e_1^d, e_2^d) cannot be a best response to μ_B^* which has Blotto playing the open lower bounds of $e_1^\# \in \Psi_1\left(T_i^e\right)$ with positive probability. Enemy could increase his payoff by playing (e_1, e_2). Given equilibrium interchangeability and the fact that no allocation $(e_1^d, e_2^d) \in S$ could be a best response to μ_B^*, **Deviation 1** cannot happen in any Nash equilibrium.

This only leaves two possible types of deviations by Enemy: He could play with mass other than $\frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}$ over some T_i^e **(Deviation 3)** and/or he could play with mass over a region S where $\forall (e_1^d, e_2^d) \in S, \quad \forall (e_1, e_2) \in T^e$:

$$e_1^d > e_1 \; or \; e_2^d > e_2$$

$$S \cap T^e = \emptyset$$

(Deviation 2). Given the bounds of the T_i^e's it is easy to show that any such region S must be within the set of points D_i^e, indexed by $i = 1, 2, \ldots, n-1$, where

$$D_i^e \equiv \left\{(e_1, e_2):\; \left(e_1 \geq h^{n-i}(E_1),\; e_2 \geq p^i(E_2),\; and\; e_2 \leq g\left(e_1\right)\right)\right\}$$

Given Lemma 5 there are no other allocations for which **Deviation 2** holds.

We simultaneously prove that neither of the latter two deviations is possible. Consider a T_i^e and D_i^e and a deviating Enemy strategy μ_E^d that forms an Nash equilibrium with any $\mu_B \in \Omega^B$. Consider some $i \in \{1, 2, 3, ..., n-1\}$. Assume that

$$\forall k = 1, 2, \dots, i-1, \quad \mu_E^d(D_k^e) = 0 \; and \; \mu_E^d(T_k^e) = \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j} \tag{43}$$

In other words, there has not "yet" been a **Deviation 2** or **Deviation 3**.

Suppose the mass over $\mu_E^d(T_i^e) < \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$. Given Lemma 7, equation 43 and the fact that we've ruled out **Deviation 1**, when Blotto plays $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$ (which he does with probability $\frac{w^{n-i}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$ in strategy μ_B^*) he wins Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e\right) < \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ but still wins Battlefield 2 with probability $1 - \mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$ for a total expected payoff strictly less than $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$, which is Blotto's constant expected payoff in all equilibria, a contradiction. Therefore, $\mu_E^d(T_i^e) \geq \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$.

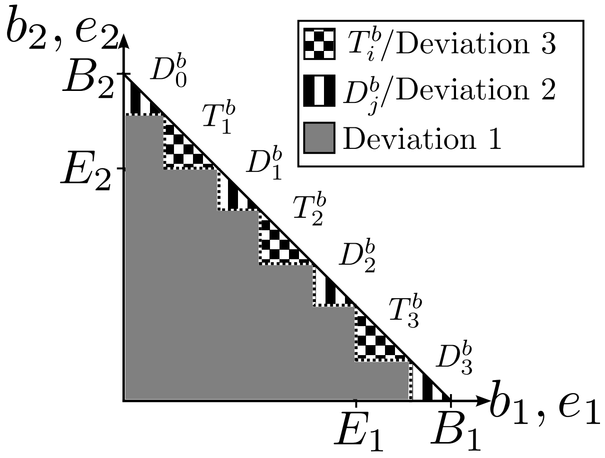
Similarly, if $\mu_E^d\left(T_i^e\right) > \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$, then when Blotto plays $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$, he wins Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e\right) > \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ but still wins Battlefield 2 with probability $1 - \mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$ for a total expected payoff strictly greater than his constant equilibrium payoff, $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$, a contradiction. Therefore, $\mu_E^d(T_i^e) = \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$.

Now suppose $\mu_E^d(D_i^e) > 0$. Now, when Blotto plays $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$ (which he does with probability $\frac{w^{n-i}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$ in strategy μ_B^*) he expects to win on Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e \cup D_i^e\right) > \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ and expects to win on Battlefield 2 with probability $1 - \mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore his total expected payoff is strictly greater than $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, another contradiction. Therefore, $\mu_E^d(D_i^e)$ must equal zero.

As the above analysis holds for all $i = 1, 2, \dots, n-1$, the mass over all such T_i^e and D_i^e must equal $\frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$ and 0, respectively. The remaining mass of $\frac{w^{n-1}}{\sum\limits_{j=0}^{n-1} w^j}$ must then be distributed over the only region left, T_n^e. Therefore, μ_E^d satisfies Property 1e. We've already discussed why it must also satisfy Property 2e. Therefore, $\mu_E^d \in \Omega^E$, and there can be no Enemy strategies which are a part of some Nash equilibrium which do not satisfy our characterization.

We have ruled out any potential Enemy strategies that deviate from our characterization of possible Nash equilibrium Enemy strategies. The proof that we have characterized the

Fig. 13: Blotto Deviations in Region 3



complete set of Blotto Nash equilibrium strategies proceeds much the same way as the proof of completeness for Enemy's strategies. However, due to the open boundaries of the T_i^e's, the proof requires additional consideration.

Lemma 12 *Any Blotto strategy which is a part of some Nash equilibrium is in Ω^B.*

Proof Suppose not. Then there exists at least one Blotto strategy which is a part of some Nash equilibrium that does not satisfy properties 1b and 2b. Call such a strategy μ_B^d We've already shown how a strategy that satisfied property 1b, but violated property 2b would give Enemy an allocation offering a payoff higher than his constant equilibrium payoff. So, any uncharacterized Blotto strategy which is part of some Nash equilibrium must violate property 1b. Property 1b specifies that Blotto must only play in his T_i^b's and provides the probability of play in each. There are two ways Blotto could violate this property: He could sometimes play outside of T^b, or his μ_B could assign an incorrect probability to some T_i^b. We break the former down into two separate deviations. The first type of deviation we consider **(Deviation 1)** are where Blotto mixes over allocations that send weakly less to both battlefields than some non-deviating allocation. Formally, a strategy, $\mu_B^{d_1}$, exhibits **Deviation 1** if the following conditions hold for some S :

$$\mu_B^{d_1}(S) > 0 \tag{44}$$

$$S \cap T^b = \emptyset \tag{45}$$

$$\forall (b_1^d, b_2^d) \in S, \quad \exists (b_1, b_2) \in T^b \quad s.t. \quad b_1^d \leq b_1 \quad and \quad b_2^d \leq b_2 \tag{46}$$

The second condition implies that one of the two inequalities in the third condition always holds strictly.

The next type of deviation we consider **(Deviation 2)** are the remaining feasible allocations outside of T^b. Specifically, these are the allocations that are outside T^b, and, relative to any allocation in T^b, send strictly more to one battlefield. Formally a strategy, $\mu_B^{d_2}$, exhibits **Deviation 2** if the following conditions hold for some S:

$$\mu_B^{d_2}(S) > 0 \tag{47}$$

$$S \cap T^b = \emptyset \tag{48}$$

$$\forall (b_1^d, b_2^d) \in S, (b_1, b_2) \in T^b \quad either \quad b_1^d > b_1 \quad or \quad b_2^d > b_2 \tag{49}$$

Consider the following sets of allocations:

$$D_0^b \equiv \{(b_1, b_2): (b_1 \geq 0), (b_2 > f(h^{n-1}(E_1))), (b_2 \leq f(b_1))\} \tag{50}$$

$$D_i^b \equiv \{(b_1, b_2): (b_1 > f^{-1}(p^{i-1}(E_2))), (b_2 > f(h^{n-i-1}(E_1))), (b_2 \leq f(b_1))\} \; \forall i = 1, ..., n-1 \tag{51}$$

$$D_n^b \equiv \{(b_1, b_2): (b_1 > f^{-1}(p^{n-1}(E_2))), (b_2 \geq 0), (b_2 \leq f(b_1))\} \tag{52}$$

Note that equations 50 and 52 are implied by 51 with the exception of the weak inequalities at the zero bounds. A strategy, $\mu_B^{d_2}$, satisfying **Deviation 2** must allocate some mass over at least one of the D_i^b's as, given Lemma 5, these are the only regions where conditions 47-49 hold. See Figure 13 for a graph of the D_i's.

The last type of deviation we consider **(Deviation 3)** is simply where Blotto plays inappropriate mass over one of his T_i^b's. Formally, a strategy $\mu_B^{d_3}$ exhibits **Deviation 3** if the following condition holds for at least one of Blotto's T_i^b's:

$$\mu_B^{d_3}\left(T_i^b\right) \neq \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}.$$

Because of Lemma 10 (Equilibrium Interchangeability), any Blotto strategy from any Nash equilibrium must form a Nash equilibrium with any Enemy strategy from Ω^E. Specifically, we consider the following sequence of Enemy strategies: For any $k = 1, 2, ...$ let μ_E^k be the strategy where in each T_i^e Enemy plays points

$$\left(g^{-1}\left(p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)\right), p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)\right)$$

and

$$\left(h^{n-i}\left(E_1 - (n+1-i)\frac{\epsilon}{k}\right), g\left(h^{n-i}\left(E_1 - (n+1-i)\frac{\epsilon}{k}\right)\right)\right)$$

with probability $\frac{w^{i-1}}{2\cdot\sum\limits_{j=0}^{n-1} w^j}$. This implies that no other allocations are in the support of μ_E^k since $2\cdot\sum_{i=1}^{n} \frac{w^{i-1}}{2\cdot\sum\limits_{j=0}^{n-1} w^j} = 1$. For ϵ sufficiently small μ_E^1 (and any other μ_E^k) satisfies properties 1e and 2e.

Intuitively, this is a sequence of strategies that has Enemy randomizing over allocations on his resource constraint, arbitrarily close to the corners in his T_i^e's. In other words, for any T_i^e we will be able to find some μ_E^k where Enemy plays arbitrarily close to each intersection of his resource constraint and either of the other two bounds for T_i^e with strictly positive probability.

To see that any μ_E^k is in fact in Ω^E, consider the following. So long as ϵ is sufficiently small, the points from μ_E^1, $\left(g^{-1}\left(p^{i-1}\left(E_2 - i\frac{\epsilon}{1}\right)\right), p^{i-1}\left(E_2 - i\frac{\epsilon}{1}\right)\right)$ and $\left(h^{n-i}\left(E_1 - (n+1-i)\frac{\epsilon}{1}\right), g\left(h^{n-i}\left(E_1 - (n+1-i)\frac{\epsilon}{1}\right)\right)\right)$, will be in T_i^e. These points are a small (note the $i\epsilon$ or $(n+1-i)\,\epsilon$ terms) distance from the intersection with their respective boundary of T_i^e. As they are a small distance towards the interior, Property 1e holds (for sufficiently small ϵ). Since these points will simply get closer to the boundary as k increases, (yet never touch it) Property 1e will hold for any μ_E^k.

The $\frac{\epsilon}{k}$ terms are multiplied by i and $(n+1-i)$ to ensure that property 2e holds. Note that full expenditure deviating play by Blotto of $\left(f^{-1}\left(p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)\right), p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)\right)$ will exactly match the $p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)$ Enemy sometimes plays on Battlefield 2, but will not increase his payoff on Battlefield 1 as $f^{-1}\left(p^{i-1}\left(E_2 - i\frac{\epsilon}{k}\right)\right)$ is strictly less than $f^{-1}\left(p^{i}\left(E_2 - (i+1)\frac{\epsilon}{k}\right)\right)$ (from T_{i+1}^b). So, Blotto's expected payoff will not increase relative to play in T_i^b. Similar analysis on other potential full expenditure Blotto deviations ensures Property 2e holds.

Now we are ready to start considering Blotto's potential deviations. **Deviation 1** has Blotto mix over allocations which send weakly less to both battlefields than some allocation in some T_i^b. Since these deviating allocations are not themselves in T_i^b they must send strictly less to at least one battlefield. Suppose these conditions hold for some set of allocations S and let $b^d = \left(b_1^d, b_2^d\right) \in S$ satisfy **Deviation 1** relative to a $\left(b_1^i, b_2^i\right) \in T_i^b$. Suppose b^d sends strictly less to Battlefield 1, or, by Lemma 5, $b_1^d < h^{n-i}(E_1)$ (and $b_2^d \leq f\left(h^{n-i}\left(E_1\right)\right)$). There exists some k^* where $b_1^d < h^{n-i}\left(E_1 - (n+1-i)\frac{\epsilon}{k^*}\right) < h^{n-i}\left(E_1\right)$. Now, we know $\left(b_1^d, b_2^d\right)$ cannot be a best response to $\mu_E^{k^*}$. Blotto could play $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$, but he plays $\left(b_1^d, b_2^d\right)$ which strictly lowers his probability of winning on Battlefield 1 when Enemy plays $\mu_E^{k^*}$, and it does so without increasing Blotto's probability of winning on Battlefield 2. Therefore $\left(b_1^d, b_2^d\right)$ provides a strictly lower payoff and cannot be a best response. Similar logic applies to $\left(b_1^d, b_2^d\right)$ that send strictly less to Battlefield 2 (or where $b_1^d \leq f^{-1}\left(p^{i-1}\left(E_2\right)\right)$ and $b_2^d < p^{i-1}(E_2)$). Therefore, all allocations which could be randomized over in **Deviation 1** are not best responses to some μ_E^k. Therefore, no equilibrium Blotto strategy exhibits **Deviation 1**.

Consider a deviating Blotto strategy μ_B^d that forms a Nash equilibrium with any $\mu_E \in \Omega^E$. All allocations in D_0^b and D_n^b are not best responses to certain Nash equilibrium Enemy strategies. For instance, take an allocation $\left(b_1^d, b_2^d\right) \in D_0^b$. Blotto is increasing his Battlefield 2 allocation while reducing his Battlefield 1 allocation relative to T_1^b. However, in T_1^b Blotto was already guaranteeing victory on Battlefield 2 so this cannot be payoff improving. Clearly $b_1^d < h^{n-1}(E_1)$. We can find some $k^* \in \mathbb{N}$ such that $b_1^d < h^{n-1}\left(E_1 - (n+1-1)\frac{\epsilon}{k^*}\right) < h^{n-1}(E_1)$. As Enemy plays $h^{n-1}\left(E_1 - (n+1-1)\frac{\epsilon}{k^*}\right)$ on Battlefield 1 with positive probability in $\mu_E^{k^*}$, $\left(b_1^d, b_2^d\right)$ must provide Blotto with a strictly lower payoff than $\left(h^{n-1}(E_1), f\left(h^{n-1}(E_1)\right)\right)$, which also guarantees victory on Battlefield 2. Therefore, $\mu_B^d\left(D_0^b\right) = 0$. Similar logic implies that $\mu_B^d\left(D_n^b\right) = 0$.

We now simultaneously prove that neither **Deviation 2** nor **Deviation 3** is possible in a Nash equilibrium. Consider some $i \in \{1, 2, 3, ..., n-1\}$. Assume that

$$\forall j = 1, 2, \ldots, i-1, \quad \mu_B^d(D_j^b) = 0 \; and \; \mu_B^d(T_j^b) = \frac{w^{n-j}}{\sum\limits_{l=0}^{n-1} w^l} \tag{53}$$

In other words, there has not "yet" been a **Deviation 2** or **Deviation 3**.

Consider possible versions of **Deviation 3** for T_i^b. First suppose, $\mu_B^d\left(T_i^b\right) > \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$. Given equation 53 and the fact that we've already ruled out **Deviation 1**, when Enemy plays in T_i^e his probability of winning on Battlefield 1 is $\mu_B^d\left(T_1^b \cup ... \cup T_{i-1}^b\right) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$.[25] However, his probability of winning on Battlefield 2 is $1 - \mu_B^d(T_1^b \cup ... \cup T_i^b) < \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore, Enemy's total expected payoff is then strictly less than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$ which is his constant equilibrium payoff, a contradiction. Similar logic applies to the case where $\mu_B^d\left(T_i^b\right) < w^{n-i}$ and implies that his expected payoff would be greater than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, a contradiction. Therefore, $\mu_B^d\left(T_i^b\right) = \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$.

Now consider a possible **Deviation 2**. Specifically, $\mu_B^d\left(D_i^b\right) > 0$. Note that all Battlefield 2 allocations in D_i^b are strictly greater than $f\left(h^{n-i-1}\left(E_1\right)\right)$. Define $S_i\left(\delta\right) = \left\{(b_1, b_2) : \; (b_1, b_2) \in D_i^b \; and \; b_2 \geq f\left(h^{n-i-1}\left(E_1\right)\right) + \delta\right\}$. We are then assured that $\exists \delta > 0$ sufficiently small that $\mu_B^d\left(S_i\left(\delta\right)\right) > 0$. Then we are also assured that $\exists k^* \in \mathbb{N}$ such that

[25] $\mu_B^d\left(\emptyset\right) = 0$ for $i = 1$.

$f\left(h^{n-i-1}\left(E_1\right)\right)+\delta > g\left(h^{n-i}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right)\right)$.[26] Note that Enemy plays $\left(h^{n-i}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right), g\left(h^{n-i}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right)\right)\right)$ with positive probability in $\mu_E^{k^*}$. When he does so the probability that he wins on Battlefield 1 is $\mu_B(T_1^b \cup ... \cup T_{i-1}^b) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, but the probability he wins on Battlefield 2 is weakly less than $1-\mu_B(T_1^b \cup ... \cup T_{i-1}^b \cup S_i(\delta)) < \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore his expected payoff is strictly less than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, a contradiction.

This analysis holds for all $i = 1, ..., n-1$. Therefore, given Lemma 5 and the definitions of the T_i^b's, D_i^b's, and **Deviation 1**, we have now determined the mass over sets containing all feasible Blotto allocations other than those in T_n^b (see Figure 13 for a graphical aide). Thus, $\mu_B^d\left(T_n^b\right) = \frac{1}{\sum\limits_{j=0}^{n-1} w^j}$, the remaining mass. Therefore, μ_B^d does not violate Property 1b. Since we've already shown it can't violate Property 2b, $\mu_B^d \in \Omega^B$.

Recall **Theorem 3**: The complete set of Nash Equilibria of any two battlefield Colonel Blotto game is the set of pairs $\{\mu_B, \mu_E\}$ such that $\mu_B \in \Omega^B$ and $\mu_E \in \Omega^E$.

Proof The theorem follows directly from Lemmas 10, 11, and 12.

B Method of Equilibrium Strategy Construction

B.1 Blotto Construction

Here we will demonstrate how to construct an equilibrium Blotto strategy from any continuous CDF whose range is $[0,1]$ over the domain of b_1 within T_1^b.

Let $z_1\left(b_1\right)$ be any continuous, strictly increasing function over the domain $\left[h^{n-1}\left(E_1\right), f^{-1}\left(E_2\right)\right]$ such that:

$$z_1\left(b_1\right) = 0 \; if \; b_1 \leq h^{n-1}\left(E_1\right) \tag{54}$$

$$z_1\left(b_1\right) = 1 \; if \; b_1 \geq f^{-1}\left(E_2\right). \tag{55}$$

So, z_1 is a continuous, strictly increasing function with a range of $[0,1]$ over the domain of b_1 within T_1^b. We let this represent a CDF of b_1 values that Blotto might play within T_1^b. Now $\forall i = 2, ..., n$ we iteratively define:

$$z_n\left(h^{-(n-1)}\left(b_1\right)\right) \equiv ...z_i\left(h^{-(i-1)}\left(b_1\right)\right)... \equiv z_2\left(h^{-(2-1)}\left(b_1\right)\right) \equiv z_1\left(b_1\right) \tag{56}$$

Each z_i represents a CDF over b_1 values that Blotto might play within T_i^b. Note that by equation 26 the lower and upper bounds of b_1 values in $\Omega\left(T_i^b\right)$ are $h^{n-i}(E_1)$ and $f^{-1}\left(p^{i-1}(E_2)\right)$ respectively. Then, by equations 54, and 56:

$$z_i\left(h^{n-i}\left(E_1\right)\right) = z_i\left(h^{-(i-1)}\left(h^{n-1}\left(E_1\right)\right)\right) = z_1\left(h^{n-1}\left(E_1\right)\right) = 0.$$

Note that by definition we have that $f^{-1}\left(p^j\left(x\right)\right) = h^{-j}\left(f^{-1}\left(x\right)\right)$. Therefore, by equations 55, and 56:

$$z_i\left(f^{-1}\left(p^{i-1}(E_2)\right)\right) = z_i\left(h^{-(i-1)}\left(f^{-1}\left(E_2\right)\right)\right) = z_1\left(f^{-1}\left(E_2\right)\right) = 1.$$

So, given this construction method, z_i equals zero at the lower bound of b_1 values in $\Psi_1\left(T_i^b\right)$ and equals 1 for the upper bound of b_1 values in $\Psi_1\left(T_i^b\right)$. Also note that all z_i are strictly increasing functions over $\Psi_1\left(T_i^b\right)$ as $h^{-(i-1)}$ and z_1 are strictly increasing functions. Therefore, each z_i is a CDF over b_1 values that Blotto might play within $\Psi_1\left(T_i^b\right)$.

[26] Note, $g\left(h^{n-i}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right)\right) = g\left(h\left(h^{n-i-1}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right)\right)\right) = f\left(h^{n-i-1}\left(E_1-(n+1-i)\frac{\epsilon}{k^*}\right)\right)$.

Proposition 13 *For any z_1, the following mixed strategy, μ_B^*, is in Ω^B : Blotto chooses $i \in \{1, 2, ..., n\}$ with the probability he chooses any particular i being given by*

$$\frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j}.$$

He then randomly chooses a b_1 within $\Psi_1\left(T_i^b\right)$ according to the CDF z_i. He then allocates $(b_1, f(b_1))$.

Proof This strategy trivially satisfies Property 1b. We now show that such a strategy satisfies Property 2b. For some $i \in 1, 2, ..., n-1$ consider an $x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-1}(E_2)\right)] = \Psi_1\left(T_i^b\right)$. By construction, and equation 20 $\mu_B^*\left(j_b^{x,i}\right) = z_i(x) \cdot \frac{w^{n-i}}{\sum_{j=0}^{n-1} w^j}$. Note that since under this strategy Blotto always fully expends his resources, the set of allocations such that $b_2 \geq g(x)$ is equivalent to the set of allocations such that $b_1 \leq f^{-1}(g(x))$. Therefore, given equation 21 $\mu_B^*\left(k_b^{x,i}\right) = z_{i+1}\left(f^{-1}(g(x))\right) \cdot \frac{w^{n-i-1}}{\sum_{j=0}^{n-1} w^j}$. Note that $h^{-1}(x) = f^{-1}(g(x))$. Therefore, by construction $z_{i+1}\left(f^{-1}(g(x))\right) = z_i(x)$, and $\mu_B^*\left(j_b^{x,i}\right) = \mu_B^*\left(k_b^{x,i}\right) \cdot w$. Thus Property 2b is satisfied.

B.2 Enemy Construction

Here we will demonstrate how to construct an equilibrium Enemy strategy from any continuous CDF whose range is $[0, 1]$ over $\Psi_1\left(T_1^e\right)$.

Let $v_1(e_1)$ be any continuous, strictly increasing function such that:

$$v_1(e_1) = 0 \; if \; e_1 \leq 0 \tag{57}$$

$$v_1(e_1) = 1 \; if \; e_1 \geq h^{n-1}(E_1). \tag{58}$$

So, v_1 is a continuous, strictly increasing function with a range of $[0, 1]$ over the domain of e_1 within T_1^e. We let this represent a CDF of e_1 values that Enemy might play within T_1^e. Now $\forall i = 2, ..., n$ we iteratively define:

$$v_n\left(h^{-(n-1)}(e_1)\right) \equiv ...v_i\left(h^{-(i-1)}(e_1)\right) ... \equiv v_2\left(h^{-(2-1)}(e_1)\right) \equiv v_1(e_1) \tag{59}$$

Note that by equations 28, 30, and 32 the lower and upper bounds of e_1 values within T_i^e are $f^{-1}\left(p^{i-2}(E_2)\right)$ and $h^{n-i}(E_1)$ respectively. Then, by equations 57, and 59:

$$v_i\left(h^{n-i}(E_1)\right) = v_i\left(h^{-(i-1)}\left(h^{n-1}(E_1)\right)\right) = v_1\left(h^{n-1}(E_1)\right) = 1.$$

Note that by definition we have that $f^{-1}\left(p^j(x)\right) = h^{-j}\left(f^{-1}(x)\right)$. Therefore, by equations 58, and 59:

$$v_i\left(f^{-1}\left(p^{i-2}(E_2)\right)\right) = v_i\left(h^{-(i-2)}\left(f^{-1}(E_2)\right)\right) = v_1\left(h\left(f^{-1}(E_2)\right)\right)$$

$$= v_1\left(g^{-1}\left(f\left(f^{-1}(E_2)\right)\right)\right) = v_1(0) = 0.$$

So, given this construction method, v_i equals zero at the lower bound of $e_1$values in $\Psi_1\left(T_i^e\right)$ and equals 1 for the upper bound of e_1 values in $\Psi_1\left(T_i^e\right)$. Also note that all v_i are strictly increasing over $\Psi_1\left(T_i^e\right)$ as $h^{-(i-1)}$ and v_1 are strictly increasing functions. Therefore, each v_i is a CDF over e_1 values that Enemy might play within $\Psi_1\left(T_i^e\right)$.

Proposition 14 *For any v_1, the following mixed strategy, μ_E^*, is in Ω^e : Enemy chooses $i \in \{1, 2, ..., n\}$ with the probability he chooses any particular i being given by*

$$\frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}.$$

He then randomly chooses a e_1 within $\Psi_1 \left(T_i^e\right)$ according to the CDF v_i. He then allocates $(e_1, g(e_1))$.

Proof This strategy trivially satisfies Property 1e. We now show that such a strategy satisfies Property 2e. For some $i \in 1, 2, ..., n-1$ consider an $x \in \left(f^{-1}\left(p^{i-1}(E_2)\right), h^{n-i-1}(E_1)\right) = \Psi_1\left(T_{i+1}^e\right)$.[27] By construction, and equation 22 $\mu_E^*\left(j_e^{x,i}\right) = v_{i+1}(x) \cdot \frac{w^i}{\sum_{j=0}^{n-1} w^j}$. Note that since under this strategy Enemy always fully expends his resources, the restriction that $e_2 \geq f(x)$ is equivalent to the restriction that $e_1 \leq g^{-1}(f(x))$. Therefore, given equation 23 $\mu_E^*\left(k_e^{x,i}\right) = v_i\left(g^{-1}(f(x))\right) \cdot \frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}$. Note that $h(x) = g^{-1}(f(x))$. Therefore, by construction

$$v_{i+1}(x) = v_{i+1}\left(h^{-1}(h(x))\right) = v_i(h(x)) = v_i\left(g^{-1}(f(x))\right),$$

and $\mu_E^*\left(j_e^{x,i}\right) = \mu_E^*\left(k_e^{x,i}\right) \cdot w$. Thus Property 2e is satisfied and $\mu_E^* \in \Omega^e$.

[27] For $i = n-1$, $\Psi_1\left(T_{i+1}^e\right)$ technically equals $\left(f^{-1}\left(p^{i-1}(E_2)\right), h^{n-i-1}(E_1)\right]$.